TA: LEE, Yat Long Luca
Email: yllee@math.cuhk.edu.hk
Office: Room 505, AB1
Office Hour: Send me an email first, then we will arrange a meeting (if you need it).

1 Review

- A set D in a metric space (X, d) is *dense* if for all $x \in X$, there exists $r > 0$ such that $B_r(x) \cap D \neq \emptyset$.
- Equivalently, D is dense in X if the closure of D is X, i.e., $\overline{D} = X$.
- A set E in (X, d) is *nowhere dense* if $\overline{E}$ has empty interior.
- A set in (X, d) is of *first category* if it can be expressed as a countable union of nowhere dense set.
- A set in (X, d) is of *second category* if it is not of first category.
- A set in (X, d) is called *residual* if its complement is of a first category.

Theorem 4.9 (Baire Category Theorem) In a complete metric space, the countable union of nowhere dense sets has empty interior. Equivalently, all residual sets are dense.

Remark: Nowhere dense set is defined such that its closure has empty interior. If the set is closed, then the above statement require only empty interior as the closure of a closed set is the closed set itself. I.e., countable union of closed set with empty interior has empty interior.

Theorem 4.9' (Baire Category Theorem) Let (X, d) be a complete metric space and $\{G_n\}$ be a sequence of open, dense subsets in X. Then the set $E = \bigcap_{n=1}^{\infty} G_n$ is dense.

Exercise 1

Source: Previous HW Problem

Use Baire category theorem to show that transcendental numbers are dense in the set of real numbers.

Solution:
Recall that a number $a \in \mathbb{R}$ is called *algebraic* if it is a root of a polynomial with integer coefficients, i.e., for some nonzero $p \in \mathbb{Z}[x]$, we have $p(a) = 0$, and is called *transcendental* if otherwise.

Let $\mathcal{A}$ and $\mathcal{T}$ be the set of all algebraic and transcendental numbers in $\mathbb{R}$, then $\mathbb{R} = \mathcal{A} \sqcup \mathcal{T}$. Recall that $\mathcal{A}$ is countable, then let $\mathcal{A}_n = \{a_1, ..., a_n\}$, such that $\bigcup_n \mathcal{A}_n = \mathcal{A}$ and hence

$$\mathcal{T} = \mathbb{R} \setminus \bigcup_n \mathcal{A}_n = \bigcap_n (\mathbb{R} \setminus \mathcal{A}_n).$$

However, $\mathbb{R} \setminus \{a_1, ..., a_n\}$ is dense, and open. Hence, $\mathcal{T}$ is dense by Baire category theorem.

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Exercise 2

Source: Royden and Fitzpatrick

Let $\mathcal{F}$ be a family of continuous real-valued functions on a complete metric space X that is pointwise bounded, i.e., for each $x \in X$, there is a constant M_x such that

$$|f(x)| \leq M_x, \text{ for all } f \in \mathcal{F}.$$

Then there is a nonempty open subset U of X on which $\mathcal{F}$ is uniformly bounded in the sense that there is a constant M such that

$$|f| \leq M \text{ on } U \text{ for all } f \in \mathcal{F}.$$

Solution:
For each n, define $E_n := \{x \in X : |f(x)| \leq n, \text{ for all } f \in \mathcal{F}\}$. E_n is closed, since f is continuous. Moreover, since $\mathcal{F}$ is pointwise bounded, for each $x \in X$, there is an n such that $|f(x)| \leq n$ for all $f \in \mathcal{F}$, i.e., $x \in E_n$. Hence,

$$X = \bigcup_{n=1}^{\infty} E_n.$$

Since X is complete, then Corollary 4.10 from the lecture notes implies that at least one of the E_n's has a nonempty interior. So, we can choose an n for which E_n contains an open ball $B(x, r)$. Hence, we obtain that on $B(x, r)$, all $f \in \mathcal{F}$ is bounded by n. Therefore, the theorem is proved by taking $U = B(x, r)$ and $M = n$.

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Exercise 3

This exercise is a corollary of the Baire category theorem.

Source: Royden and Fitzpatrick

Let X be a complete metric space and $\{F_n\}_{n=1}^{\infty}$ a countable collection of closed subsets of X. Then $\bigcup_{n=1}^{\infty} \partial F_n$ has empty interior.

Solution:
Recall the following definitions

- A point $x \in E$ is called an *interior point* of E if there is a $r > 0$ such that $B(x, r) \subset E$.
- The collection of interior points of E is the *interior* of E.
- A point $x \in X \setminus E$ is an *exterior point* of E if there is a $r > 0$ such that $B(x, r) \subset X \setminus E$.
- The collection of exterior points of E is the *exterior* of E.

- A point $x \in X$ is a *boundary point* of E if there is a $r > 0$ such that $B(x, r)$ contains points in the interior of E and the exterior of E.
- The collection of boundary points of E is the boundary of E, denoted by ∂E.
- An equivalent definition of ∂E would be $\partial E = \overline{E} \cap \overline{X \setminus E}$

One can see that ∂E has empty interior, since for all $x \in \partial E$, and all $r > 0$, $B(x, r) \not\subset \partial E$. One sees that ∂E is also closed, since it is the intersection of two closed sets.

Then $\{\partial F_n\}$ is a collection of closed sets with empty interior. By Baire category theorem, $\bigcup_n \partial F_n$ has empty interior.

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Exercise 4

Source: Previous HW and Leon's Tutorial notes

A function $f \in C[0, 1]$ is called *non-monotonic* if for all closed subintervals $J \subset I := [0, 1]$ of positive length, f is not monotonic on J. Show that $\mathcal{N} := \{f \in C(I) : f \text{ is non-monotonic}\}$ is dense in $C(I)$.

Solution:
By Baire category theorem, it suffices to show that $\mathcal{N}$ is residual.

Let $A := \{(x, n) \in I \times \mathbb{N} : x \in \mathbb{Q}, x \neq 0, 1\}$, then A is countable. For all $(x, n) \in A$, we define

- $\mathcal{E}_{x,n} = \{f \in C(I) : \text{ for all } y \in \overline{B_{\frac{1}{n}}(x)} \cap I, (f(y) - f(x))(y - x) \geq 0\}$ i.e., f is increasing.
- $\mathcal{F}_{x,n} = \{f \in C(I) : \text{ for all } y \in \overline{B_{\frac{1}{n}}(x)} \cap I, (f(y) - f(x))(y - x) \leq 0\}$ i.e., f is decreasing.

Note that $f \notin \mathcal{N} \iff f$ is not non-monotonic $\iff$ there exists $J \subset I$ as above such that f is monotonic over $J \iff$ there exists $(x, n) \in A$ s.t. $f \in \mathcal{E}_{x,n} \cup \mathcal{F}_{x,n}$. Hence, we deduced that $C(I) \setminus \mathcal{N} = \mathcal{E}_{x,n} \cup \mathcal{F}_{x,n}$.

Following the idea, we need to show that $\mathcal{E}_{x,n} \cup \mathcal{F}_{x,n}$ is nowhere dense. That is, we want to show that $\mathcal{E}_{x,n}$ and $\mathcal{F}_{x,n}$ are nowhere dense.

Show that $\mathcal{E}_{x,n}$ is nowhere dense.

Step 1 - $\mathcal{E}_{x,n}$ is closed.

For all converging sequence $\{f_k\} \subset \mathcal{E}_{x,n}$, our goals is to show that the limit of f_k, denoted by f, converges to $f \in C(I)$.

By definition and assumption, for all $k \in \mathbb{N}$, all $y \in \overline{B_{\frac{1}{n}}(x)} \cap I$, we have that

$$(f_k(y) - f_k(x))(y - x) \geq 0$$

then

$$(f(y) - f(x))(y - x) = \lim_{k \to \infty} (f_k(y) - f_k(x))(y - x) \geq 0.$$

Hence $f \in \mathcal{E}_{x,n}$.

Step 2 - $\mathcal{E}_{x,n}$ is nowhere dense.

For all $f \in E_{x,n}$, we show that for all $\varepsilon > 0$, the ball $B_\varepsilon(f) \not\subset E_{x,n}$.

By Weierstrass approximation theorem, there exists a polynomial P s.t. $P \in B_{\frac{\varepsilon}{2}}(f)$. Since $P|_I$ is C^1, it is Lipschitz continuous. We let L to be its Lipschitz constant.

For all $n \in \mathbb{N}$, define $\varphi_N : I \longrightarrow \mathbb{R}$ be the *jig-saw* function, i.e., a piecewise linear, $\frac{1}{N}$-periodic functions with slopes $\pm 2N$. Define $g_N(x) := P(x) + \frac{\varepsilon}{2}\varphi_N(x)$. Then g is continuous on I. We check that

- $g_N \in B_\varepsilon(f)$. Since

$$\|g_N - f\|_\infty = \left\|P + \frac{\varepsilon}{2}\varphi_N - f\right\| = \left\|P - f + \frac{\varepsilon}{2}\varphi_N\right\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

- $g_N \notin \mathcal{E}_{x,n}$ for some N. Since for all $y \in I$, with $y > x$,

$$\begin{aligned}(g_N(y) - g_N(x))(y-x) &= (P(y) + \frac{\varepsilon}{2}\varphi_N(y) - P(x) - \frac{\varepsilon}{2}\varphi_N(x))(y-x) \\ &= (P(y) - P(x) + \frac{\varepsilon}{2}\varphi_N(y) - \frac{\varepsilon}{2}\varphi_N(x))(y-x) \\ &\leq (L(x-y) + \frac{\varepsilon}{2}(\varphi_N(y) - \varphi_N(x)))(y-x)\end{aligned}$$

  Now we want to obtain an estimate related to the latter term. Choose $N \in \mathbb{N}$ satisfying

$$\begin{cases} N > \frac{L}{\varepsilon} \\ \frac{2i-1}{2N} \leq x < \frac{i}{N}, \text{ for some } i \in \mathbb{N}; 1 \leq i \leq N \end{cases}$$

  Choose any $y \in I$ with $x < y < \frac{i}{N}$ and $y - x \leq \frac{1}{n}$, then

$$\varphi_N(y) - \varphi_N(x) = (-2N)(y-x),$$

  hence

$$(g_N(y) - g_N(x))(y-x) \leq (L(y-x) - N\varepsilon(y-x)(y-x)) = (L - N\varepsilon)(y-x)^2 < 0.$$

  Therefore $g \notin \mathcal{E}_{x,n}$.

Thus, $\mathcal{E}_{x,n}$ is nowhere dense.

One can then verify that $\mathcal{F}_{x,n}$ is nowhere dense in a very similar manner, and the claim is thus proven.

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